Results Bounding the Non-p-Soluble Length of Finite Groups

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Abstract

Let p be a prime. Every finite group G has a normal series each of whose quotients either is p-soluble or is a direct product of nonabelian simple groups of orders divisible by p. The non-p-soluble length $\lambda_p(G)$ is defined as the number of non-p-soluble quotients in a shortest series of this kind.

We deal with the question whether, for a given prime p and a given proper group variety $\mathfrak{V}$, the non-p-soluble length $\lambda_p(G)$ of a finite group G whose Sylow p-subgroups belong to $\mathfrak{V}$ is bounded.

In joint work with Pavel Shumyatsky, we answer the question in the affirmative in some cases (working separately the case $p = 2$) for varieties of groups in which the commutators have conditions that depends on exponent conditions and Engel conditions.

Non-Soluble and Non-p-Soluble Length

Every finite group G has a normal series each of whose quotient either is soluble or is a direct product of nonabelian simple groups. In [5] the *nonsoluble length* of G, denoted by $\lambda(G)$, was defined as the minimal number of nonsoluble factors in a series of this kind: if

$$1 = G_0 \leq G_1 \leq \cdots \leq G_{2h+1} = G$$

is a shortest normal series in which for i even the quotient G_{i+1}/G_i is soluble (possibly trivial), and for i odd the quotient G_{i+1}/G_i is a (non-empty) direct product of nonabelian simple groups, then the nonsoluble length $\lambda(G)$ is equal to h. For any prime p, a similar notion of non-p-soluble length $\lambda_p(G)$ was defined by replacing "soluble" by "p-soluble" and "simple" by "simple of order divisible by p". Recall that a finite group is said to be *p-soluble* if it has a normal series each of whose quotients is either a p-group or a p'-group. We have, $\lambda(G) = \lambda_2(G)$, since groups of odd order are soluble by the Feit–Thompson theorem [3].

We show a specific normal series that allows us to obtain the non-p-soluble length of a finite group G. For this, we establish some notations.

The soluble radical of a group G, the largest normal soluble subgroup, is denoted by $R(G)$. The largest normal p-soluble subgroup is called the *p-soluble radical* and it will be denoted by $R_p(G)$.

Consider the quotient $\bar{G} = G/R_p(G)$ of G by its p-soluble radical. The socle $Soc(\bar{G})$, that is, the product of all minimal normal subgroups of $\bar{G}$, is a direct product $Soc(\bar{G}) = S_1 \times \cdots \times S_m$ of nonabelian simple groups S_i of order divisible by p. Set the following series

$$1 = G_0 \leq \Gamma_1 \leq M_1 \leq \Gamma_2 \leq M_2 \cdots \leq G$$

where Γ_i and M_i are defined recursively by

$$\frac{M_i}{\Gamma_{i-1}} = R_p\left(\frac{G}{\Gamma_{i-1}}\right) \qquad \frac{\Gamma_i}{M_i} = Soc\left(\frac{G}{M_i}\right).$$

The number of Γ_i appearing in this series is the non-p-soluble length of G.

Upper bounds for the nonsoluble and non-p-soluble length appear in the study of various problems on finite, residually finite, and profinite groups. For example, such bounds were implicitly obtained in the Hall–Higman paper [4] as part of their reduction of the Restricted Burnside Problem to p-groups.

The Problem

There is a long-standing problem on p-length due to Wilson (Problem 9.68 in Kourovka Notebook [1]): *for a given prime p and a given proper group variety $\mathfrak{V}$, is there a bound for the p-length of finite p-soluble groups whose Sylow p-subgroups belong to $\mathfrak{V}$?*

In [5] the following problem, analogous to Wilson's problem, was suggested.

Problem A. *For a given prime p and a given proper group variety $\mathfrak{V}$, is there a bound for the non-p-soluble length λ_p of finite groups whose Sylow p-subgroups belong to $\mathfrak{V}$?*

It was shown in [5] that an affirmative answer to Problem A would follow from an affirmative answer to Wilson's problem. On the other hand, Wilson's problem so far has seen little progress beyond the affirmative answers for soluble varieties and varieties of bounded exponent [4] (and, implicit in the Hall–Higman theorems [4], for (n-Engel)-by-(finite exponent) varieties). Problem A seems to be more tractable.

Results

In the sequel, we give some notations and we present some positive answers to Problem A. For instance in [5] a positive answer was obtained in the case of any variety that is a product of varieties that are either soluble or of finite exponent.

For $s = 0, 1, \ldots$ we shall denote by $\mathfrak{X}_s(e, n)$ *the variety of all groups in which δ_s^e-values are n-Engel*, where e, s and n are non-negative integers and write ${\mathfrak{X}_s}^k(e, n)$ for the product of k varieties $\mathfrak{X}_s(e, n)$.

We obtain the following theorem:

Theorem 1. *Let p be an odd prime. The non-p-soluble length of finite groups whose Sylow p-subgroups belong to ${\mathfrak{X}_s}^k(e, n)$ is bounded in terms of k, e, n and s only.*

In this kind of result, we obtain a theorem which includes the case $p = 2$.

Theorem 2. *Let $\mathfrak{X}_1(e, n)$ be the variety defined above, and let p be a prime. The non-p-soluble length of finite groups whose Sylow p-subgroups belong to $\mathfrak{X}_1(e, n)$ is bounded in terms of e and n only.*

We are trying to prove that Theorem 1 remains valid also for $p = 2$.

The following lemma was proved in [5]. It depends on the classification of finite simple groups, and it should be noted as one of the strongest tools for obtaining the above results. We need to introduce the following definition.

Let G be a finite group and $Soc(G/R_p(G)) = S_1 \times \cdots \times S_m$. The group G induces by conjugation a permutational action on the set $\{S_1, \ldots, S_m\}$. Let $K_p(G)$ denote the kernel of this action. In [5] $K_p(G)$ was given the name of the *p-kernel subgroup* of G. Clearly, $K_p(G)$ is the full inverse image in G of $\bigcap N_{\bar{G}}(S_i)$.

Lemma 1. *The p-kernel subgroup $K_p(G)$ has non-p-soluble length at most 1.*

The technique we use to prove the results is essentially using the well behavior of the length about the extensions, i.e., suppose $\lambda(N) = l_1$ and $\lambda(G/N) = l_2$ where N is a normal subgroup of a finite group G, then the non-soluble length $\lambda(G)$ is at most $l_1 + l_2$. Using this property and Lemma 1, we try, through the iteration of quotients for the kernel group, to decrease the parameters of the variety and to use inductive arguments.

References

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